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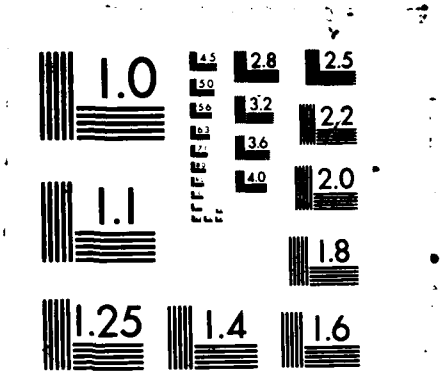

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Computational aspects of the h, p and h-p versions of the finite element method

I. Babuška and T. Scapolla

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The paper is an invited presentation at the Sixth IMAC International Symposium on Computer Methods for Partial Differential Equations, June 23-26, 1987, Lehigh University. It discusses computational complexity of various versions of the finite element method in relation to the achieved accuracy of the finite element solution.

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Abstract

The paper is an invited presentation at the Sixth IMAC International Symposium on Computer Methods for Partial Differential Equations, June 23-26, 1987, Lehigh University. It discusses computational complexity of various versions of the finite element method in relation to the achieved accuracy of the finite element solution.

Computational aspects of the h, p and h-p versions of the finite element method

Ivo Babuška (*)
Institute for Physical Science and Technology
University of Maryland, College Park, MD 20742
Terenzio Scapolla (*)
Institute for Physical Science and Technology
University of Maryland, College Park, MD 20742
and
Istituto di Analisi Numerica del CNR, 27100 Pavia, Italy

1. Introduction

There are many versions and forms of finite element methods implemented in the finite element codes. The basic architectures of the standard finite element programs are essentially very similar. Once the class of problems that can be solved has been selected (elasticity, plates and shell problems, heat transfer etc.), one of the main features is the variational form given to the problem (primary, mixed, hybrid etc.) together with the type of finite element used (conforming, non-conforming etc.).

Nowadays several hundred of finite element program systems are available, covering a wide range of class of problems. Despite the great number of codes it is very difficult to find comparison results between different programs. The main reason is due to the fact that the comparison is a very complex task involving a large number of different factors. The usual approach is to test different methods and codes on benchmark problems (see, e.g., *MacNeal* and *Harder* [14], *Robinson* and *Blackham* [20,21]). Although this approach has obviously various drawbacks, it gives good and concrete data for an effective comparison. Of course additional aspects like convergence, rate of convergence, effect of the computer technology etc., are essential as well. Different simple model problems together with a theoretical modeling are also very worthwhile.

The goal of this note is to address some results allowing basic essential comparison between finite element methods. There are three major version of the finite element method: the standard h – *version*, the p – *version* and the h – p *version*. The p– and h – p versions are a recent development. In *Szabo'* [25], *Guo* and *Babuška* [10], *Suri* [22] recent advances in the p– and h – p versions are given. For a survey of the state of the art see also *Babuška* [3]. We will try to make some comparisons between these three versions, focusing especially on their computational aspects. We will particularly analyse the relations between the accuracy of the error and the computational work. Of course any comparison is very complex, hence we will consider only some aspects of finite element method for linear problems. Finally we will consider a benchmark problem and we will see that the practical results are in good agreement with the ones based on the computational theoretical models.

(*) Work partially supported by the Office of Naval Research under Grant N-00014-85-K-0169

2. The scheme of the finite element method

The finite element method consists of few basic phases:

(a) *Topology*

By this we mean the mesh generation. The mesh has to respect the geometry of the domain under consideration and the requirement that effective and accurate solution will be obtained. The mesh generation is a very laborious part of the method also when sophisticated mesh generators are available. Very often, especially in three dimensions, the mesh reflects only the geometry.

(b) *Local stiffness matrices*

The complexity of the computation of local stiffness matrices depends on the degree of the elements and the type of hardware used (sequential or parallel).

Let us consider here computations of a C^0 quadrilateral element of degree p for a two fields (elasticity in two dimensions) problem. Using hierarchical elements (see, e.g., *Noor* and *Babuška*[4], *Szabo'* [23]), an element of degree p has Q *shape functions*, where

$$Q = 8 \quad , \quad p = 1 \tag{1}$$

$$Q = 2 \left[4p + \max\left(0, \frac{1}{2}(p-2)(p-3)\right) \right] \quad , \quad p > 1$$

The second part in the formula for Q gives the number of *internal* shape functions which are treated later by condensation procedure.

Proper programmming on a sequential machine can be made so that $O(p^4)$ operations are needed for computation. An experimenetal program in the (normalised VAX) time units leads to the computational work for the local stiffness matrices W_{LM}, where

(2) $$W_{LM} = \frac{1}{25}(2.5 + 0.032\, p^4).$$

Figure 1 shows, for different values of p, the time units needed for the computation of 25 stiffness matrices.

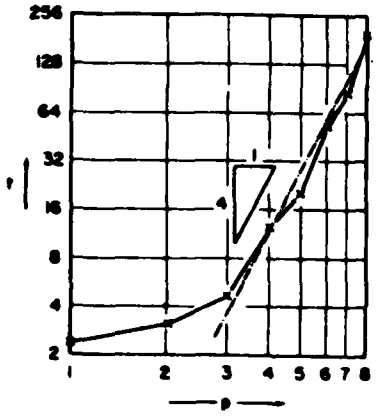


Fig.1

(c) *Assembly and elimination*

The work needed for this phase depends on the topology and the degree of the elements. The practically most complex topology in two dimensions is the case of the uniform square mesh. Assuming a decomposition with m^2 elements, the experimental program in the (normalised VAX) time units of the elimination methods leads to the following expression for the computational work W_E (in the range $1 \le p \le 8$, $1 \le m \le 5$):

(3) $$W_E = 2 + (0.0136 + 0.004\, m^3)\, p^3 .$$

Figure 2 shows the experimental time (in VAX units) as a function of p and m.

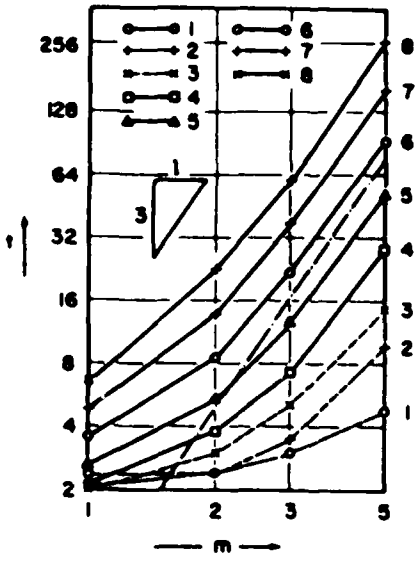


Fig.2

We recall that in the case of uniform square mesh the number N of degrees of freedom is the following:

(4) $$N = 8(m+1)^2, \quad p = 1,$$

$$N = 2[(m+1)^2 + 2m(m+1)(p-1) + m^2 \max(0, \frac{1}{2}(p-2)(p-3))], \; p > 1.$$

(d) *Postprocessing*

By this we mean computations of desired values, error control, graphics etc. This phase is often computationally very expensive. Because the cost of this phase is nearly the same for all the versions of the finite element method, we will not consider this phase.

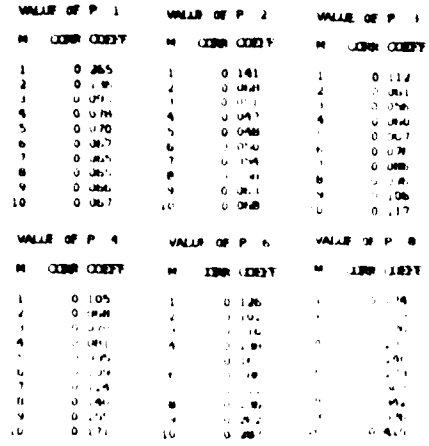
We consider as total work of computation W only the work needed for the phases (b) and (c). We have

(5) $$W = m^2 W_{LM} + W_E .$$

Sometimes different methods are compared with respect to the number N of degrees of freedom instead of computational work. Hence we define a correction coefficient

$$\rho(m,p) = \frac{W}{N} .$$

Table 1 gives ρ for various m and p.



Tab.1

3. The goal of the computations and the principles of comparisons of the method

As we said before, comparison of methods and codes is a very complex problem. It depends on many factors, one important being the man power cost needed to operate the code in practical environment. The goal of the computation is to achieve the desired data in the range of a given accuracy. We will try to assess the various versions of the finite element method and we will concentrate only on the energy norm error e_E. To get reasonably valid conclusions we will use

a- theoretical model analysis;

b- benchmark example analysis.

The main idea of the theoretical model analysis is to assume that the energy norm error can be given as an explicit function depending on both number and degree of elements. We suppose that

$$\|e\|_E \simeq \Psi(m,p) , \tag{6}$$

where $\Psi(m,p)$ is a known function. We will derive suitable expressions for the function $\Psi(m,p)$ so that it has the forms close as much as possible to the theoretical estimates for various versions of the method. Then we will study the dependence between the error $\|e\|_E$ and the computational work W. Although this approach has, of course, various shortcomings, nevertheless it can lead to valuable conclusions.

The benchmark example will be the "classical" engineering problem of a simply supported rhombic plate. We selected to present only this benchmark problem because it has different character (C^1 elements are used) and our conclusions based on plane elasticity model computations (C^0 elements) are still valid. Of course additional informations and estimates could be here used too. Nevertheless we will see that the results are in good agreement with the conclusions based upon the theoretical model for the plane elasticity.

4. Complexity of the finite element computation

The effectiveness of the method depends on the class of problem characterising the function $\Psi(m,p)$ in (6), which is a form of an estimate.

a) *The quasi uniform mesh*

Let u_0 be the exact solution of the given problem and e be the error in the $h-p$ version. Then in *Babuška* and *Suri* [6] the following estimate has been proved:

$$\|e\|_E = \frac{h^\mu}{p^{k-1}} C(k) \|u_0\|_{H^k(\Omega)} , \tag{7}$$

where h is the mesh size, $\mu = \min(p, k-1)$, Ω is the given domain, $H^k(\Omega)$ the usual Sobolev space and $C(k)$ a constant independent of h and p but depending on k and Ω. In our case we assume that $h = m^{-1}$.

Usually the function u_0 can be imbedded in various Sobolev spaces. Hence we will assume that

$$C(k) \|u_0\|_{H^k(\Omega)} \simeq \Phi(k) , \tag{8}$$

where $\Phi(k)$ is an a-priori known function. We can assume that

$$\Phi(k) = \frac{1}{(k_0 - k)^\alpha} , \tag{9}$$

with α positive, which shows that

$$u_0 \in H^{k_0-\varepsilon}(\Omega),\ \varepsilon > 0, \quad u_0 \notin H^{k_0}(\Omega).$$

This type of function $\Phi(k)$ expresses the case when the function u is *uniformly unsmooth* over the entire domain or has singular behaviour in an a-priori unknown location.

Another form we will consider is the class of analytic solutions. In this case we will assume

$$\Phi(k) = k!\, d^k , \tag{10}$$

where $d > 1$ (the coefficient d expresses the size of the natural domain of the analyticity).

Assuming the computational work W given by (5) and the function $\Phi(k)$ by (9), we show in the figures 3,4,5,6 the dependence of the accuracy of the error $\|e\|_E$ on the computational work W (in *log – log* scale) for different values of k_0 and $\alpha = 1$. The values shown are the minimal errors for integers $k = 1, 2, .., k_0 - 1$ (of course also the error in the entire range for k could be easily computed, but it would not change the conclusions). In all these figures the lines corresponding to the values $p = 1, 2, ..., 8$ are reported. On each line different values of $m = 1, 2, ...$ are considered.

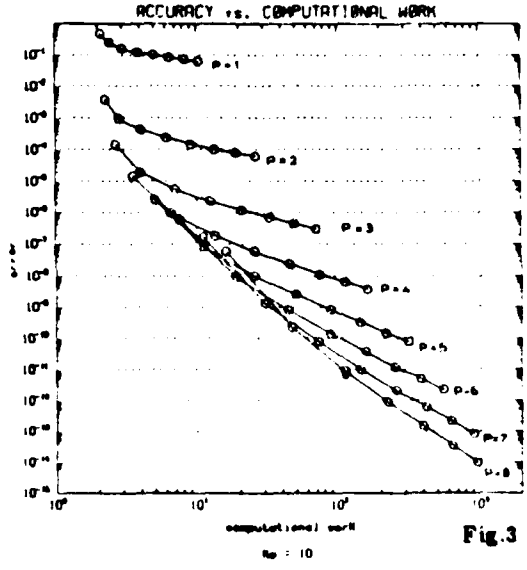


Fig.3

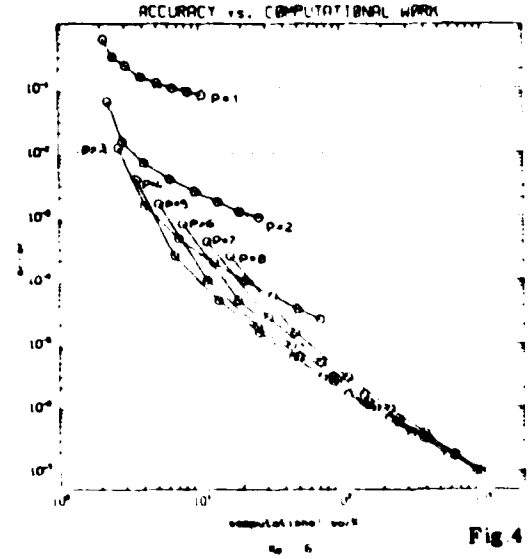


Fig.4

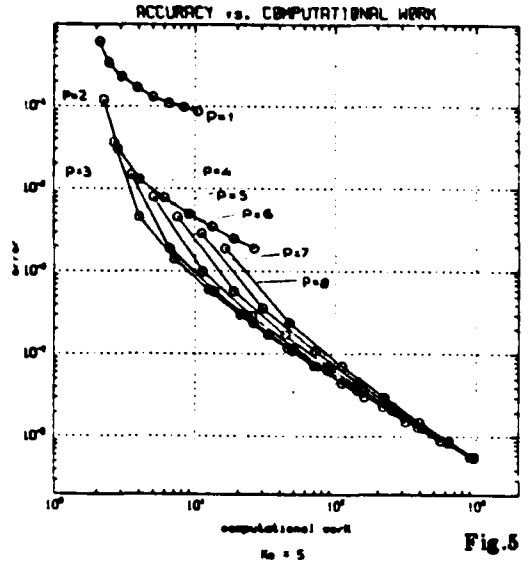


Fig.5

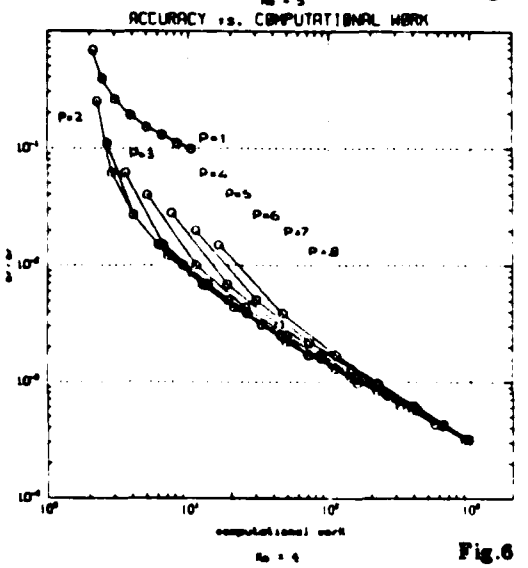


Fig.6

Let us now consider the case when $\Phi(k)$ is given by (10). The figures 7 and 8 show the behaviour of the error against the computational work for two different values of d.

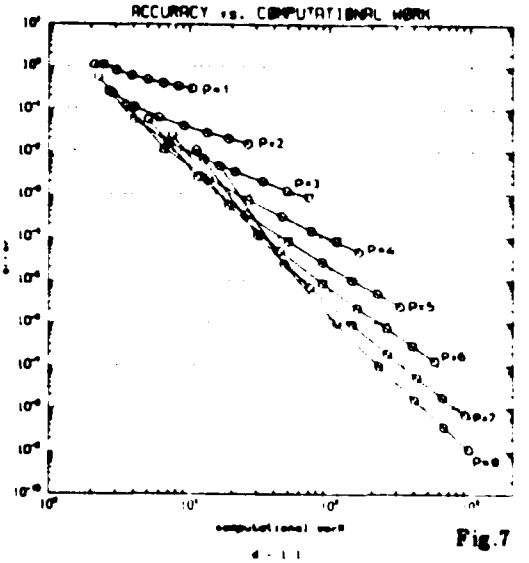


Fig.7

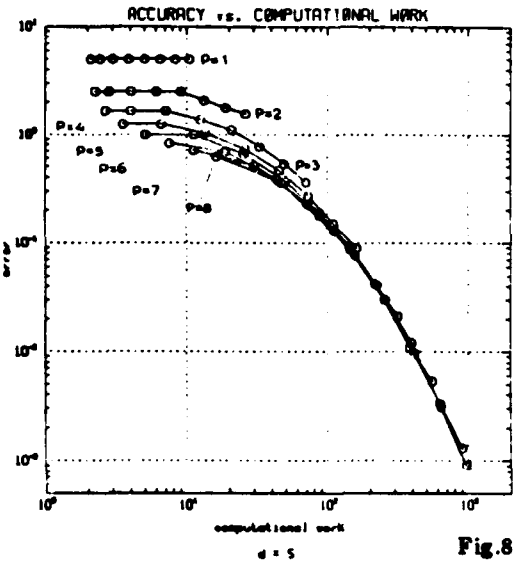


Fig.8

Let us now consider a class of problems where the solution is *singular* in a vertex of the domain. In this case the solution has the leading singularity of the type

$$u_0 = r^\alpha \underline{\phi}(\theta) , \tag{11}$$

where $r > 0$, $\alpha > 0$ and $\underline{\phi}$ is an analytic vector function in θ. Then in *Babuška* and *Suri* [6] it has been proved that

$$\|e\|_E = \min\left(m^{-\alpha}, \frac{m^{-\min(\alpha, p-\alpha)}}{p^{2\alpha}}\right) . \tag{12}$$

The figures 9,10,11 show the accuracy of the error against computational work for different values of α.

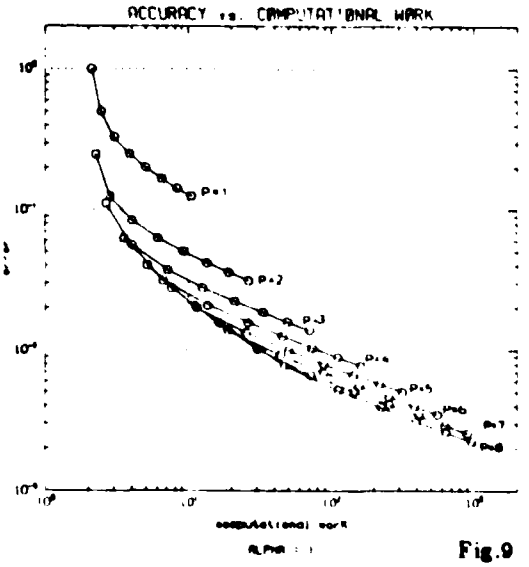


Fig.9

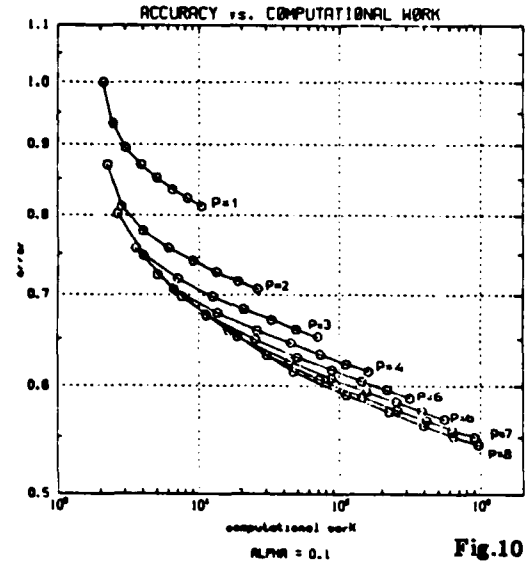


Fig.10

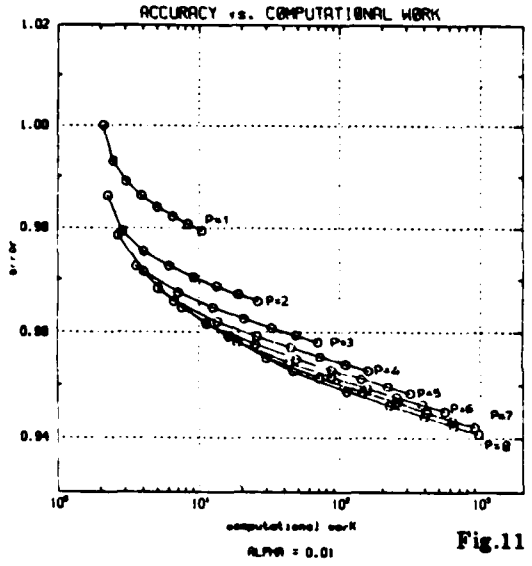


Fig.11

So far we have mentioned only decompositions of the given domain with uniform mesh. If the solution is of the type (11) and an optimal mesh is used, then the error behaviour is close to the case of the smooth solution previously described.

b) *The strongly refined mesh for the h − p version*

Here we consider the solution u_0 belonging to the space $B^2_\beta(\Omega)$ (see *Guo* and *Babuška* [11,12]). In this case the following estimate holds for the error:

$$\|e\|_E = C\, exp(-b\, N^{\frac{1}{3}}) \,. \tag{13}$$

The topology of the (strongly refined) mesh is equivalent to a rectangle (not a square). Nevertheless we will use our result for the square mesh topology (which leads to the rate $exp(-b\, N^{\frac{1}{2}})$). In this case the degree p of the elements that are used depends on m. We will use for the error the relation which leads to the above estimate and is used in [11,12]:

$$\|e\|_E = exp(-\alpha p) \quad , \quad p = [sm] \quad , \tag{14}$$

where $[sm]$ denotes the integer part of sm. The figure 12 shows the error against the work when the relation (14) is used with $\alpha = 0.5$.

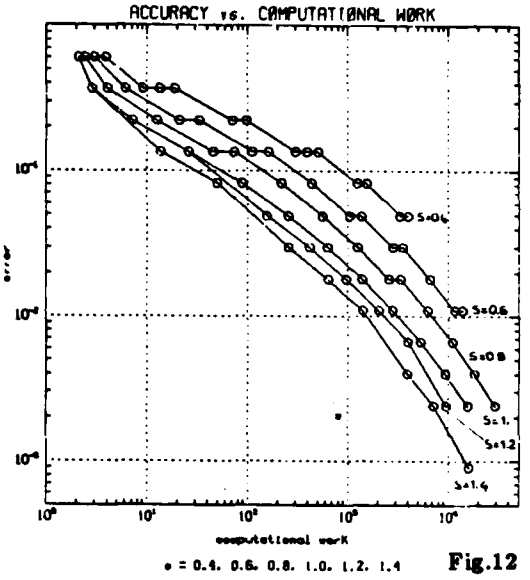


Fig.12

c) *Conclusions*

A careful analysis of the figures previously shown leads to the following conclusions:

1- Only for very low required accuracy or very unsmooth solutions (k_0, α small) the lower order elements are preferable, on a uniform mesh. However, this is not usually the practical case.

2- The increase of the quality of the solution from $p = 1$ to $p = 2$ is very significant and $p = 2$ gives sometimes reasonable errors for low but already acceptable engineering accuracy.

3- It is safer to use elements of higher degree (3-5) then lower ones for the effective computation. The use of higher degree elements in general is more robust. Let us remark that in the elasticity problem the *locking* effect (for $\nu \simeq 0.5$) is completely eliminated when $p \geq 4$ (see *Scott* and *Vogelius* [27]). This is another reason to use $p \geq 3$.

4- If the solution is smooth or the solution has singularities but the mesh is properly refined, higher order elements are preferable except for very low desired accuracy.

5- The most effective way is to combine the degree of the elements with a properly designed mesh. Of course the effectiveness depends very much on the proper design of the mesh. For questions of this type we refer to *Guo* and *Babuška* [13], *Rank* and *Babuška* [18].

6- A high practical accuracy can usually be obtained with the elements of degree $p \leq 8$ when reasonable mesh is used in all practical cases.

5. A benchmark problem: the rhombic plate

As we said in the introduction, the usual way to compare different finite element methods is to test them on a benchmark problem, where the solution is in some way explicitly known. In order to evaluate the effectiveness of the theoretical error analysis model previously introduced several tests has been performed. In particular we refer to *Babuška* and *Suri* [6] for various results related to the solution of the elasticity problem of an L-shaped domain. However, we show that our conclusions are useful for more general situations beyond the specific one from which it has been derived (of course a similar model could be made based on available results for plate bending problems). Hence we have analyzed another "classical" benchmark problem, the bending of a simply supported rhombic plate under normal loading, to evaluate the generality of our approach.

a) *The rhombic (Kirchhoff) plate problem*

The most severe test for plate elements is the so-called Morley's skew plate (see *Morley* [15,16]), which is a uniformly loaded and simply supported plate of the shape of an equilateral parallelogram (rhombus).
In Fig.13 a rhombic plate is shown together with a 2×2 mesh for triangular elements.

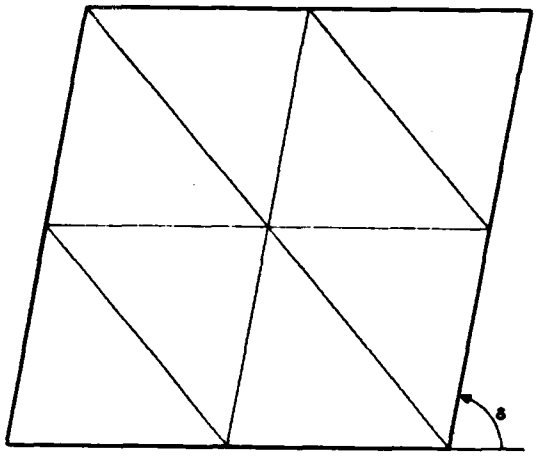


Fig.13

Because of the stress singularity at the obtuse corners of tha plate the convergence of the finite element solutions has been found to be extremely slow. The singularity depends on the angle δ and as δ decreases the singularity becomes stronger. *Robinson* [19] has compiled the results of many finite element codes for this problem. The results of most of the elemnts show a very large error for the center deflection of the plate even when very fine mesh is used. However, no indications are given about the real cost of using each element to get a desired accuracy. Our main goal is to test some elements on the rhombic plate taking into account accuracy *and* computational work.

(b) *The finite elements*

Several finite elements for plate are known. We have restricted our attention to *conforming* elements, that is elements satysfying the C^1-condition, allowing the discrete approximation space to be included in the continuous one. The main reason is that conforming elements have some good properties of monotonicity (i.e. for the energy) enabling an effective control of the error.However in many codes non-conforming elements are used.

The elements we have tested are the following:

1- reduced Hsieh-Clough-Tocher triangle (HCTR),

2- assumed stress hybrid triangle (HYBR),

3- Argyris triangle (ARGY).

The Fig.14 shows, with the usual convention for notation, the type of degrees of freedom associated with each type of element. For details we refer to [8] for HCTR, [7,17] for HYBR, [1] for ARGY.

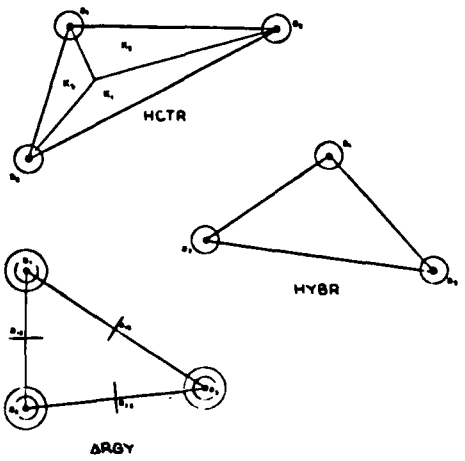


Fig.14

We only recall that the following abstract error estimates hold provided a sufficiently smoothness of the solution:

$$||e||_E \leq C\, h\, ||u||_s$$

for both HCTR and HYBR elements,

$$||e||_E \leq C\, h^4\, ||u||_s$$

for ARGY element.

(c) *The numerical results*

Numerous tests have been performed for different shapes of the plate and for different boundary conditions Here we focus our attention on the *energy error* obtained for various values of the angle δ. We have considered a plate with the following data:

side length $a = 1.0\ in$,
thickness $t = 0.01\ in$,
Young's modulus $E = 30 \times 10^6\ lb/in$,
Poisson's ratio $\nu = 0.3$
normal load $p = 1.0\ lb/in^2$.

For the plate problem we have computed the (relative) energy error, defined in the following way:

$$\|e\|_E = \left(\frac{E_E - E_C}{E_E}\right)^{\frac{1}{2}} \times 100\,,$$

where E_E, E_C represent the exact and the computed energy, respectively. The computation has been carried out using routines of the code MODULEF (see [9]) and the final linear system has been solved by Cholesky procedure. The computation was performed on an Apollo DN300 computer. We have used existing subroutines to avoid any bias on the reached conclusions.

The Fig.15,16 show the behaviour of the energy error against the total time of computation for different values of the angle δ. As expected, the error increases as the angle δ decreases. However we note that in all the situations the higher degree element ARGY gives better performances over the lower degree elements HCTR and ARGY. More important, if we compare these results with the Fig.9,10,11 we see the agreement between our model and the concrete numerical results.

To better understand the equivalence we show in Fig.17,18 the lines corresponding to degrees p=1 (HCTR and HYBR element) and p=4 (ARGY element) for values of α=0.8 and α=0.2 based on computation using relation (12) (in the same way as Fig.9,10,11 were computed). These values of the parameter describing the singularity are theoretically related to the values of δ=80 and δ=30, respectively. The direct comparison (Fig.15 and 17; Fig.16 and 18) shows a good agreement between our model and practical results.

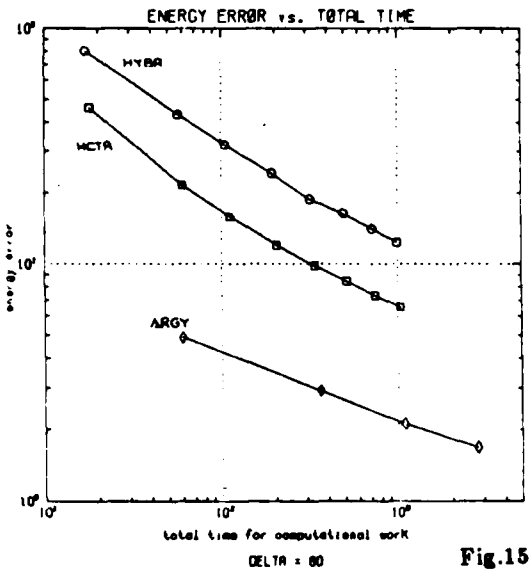
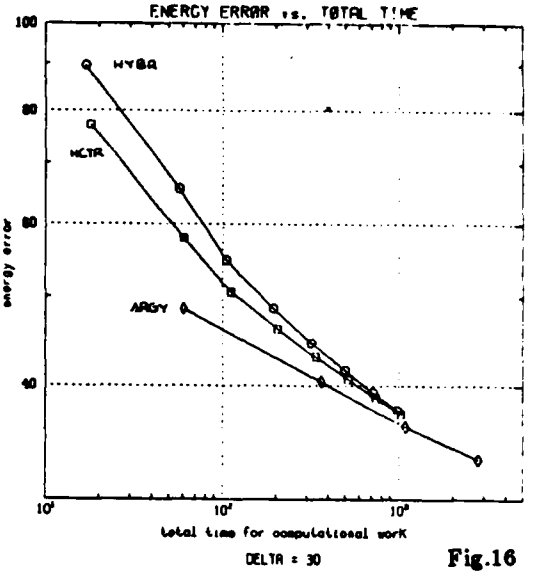


Fig.15



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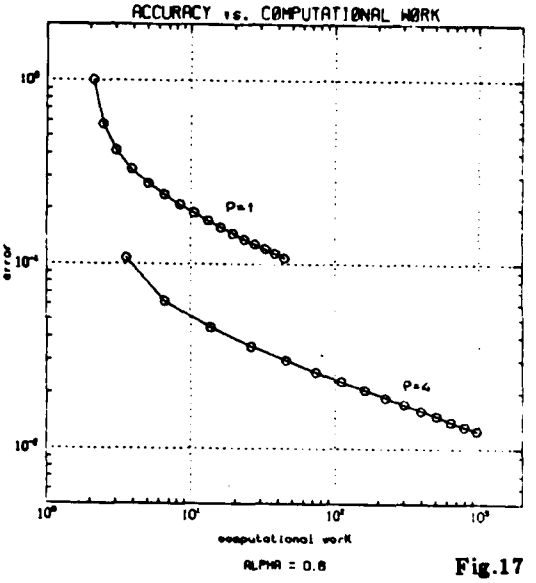

Fig.17

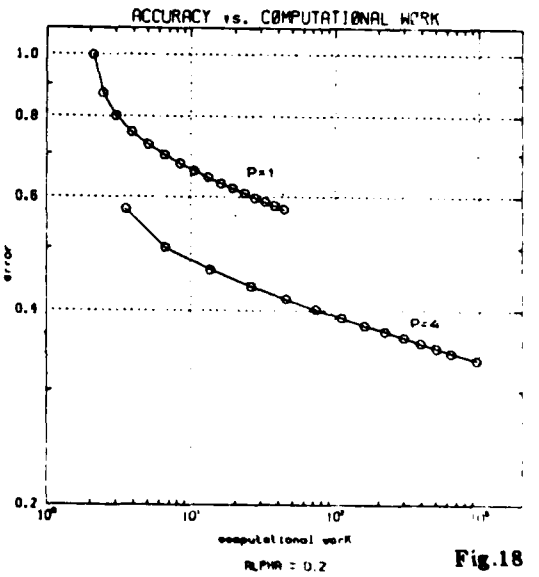

Fig.18

6. Conclusions

1- Models of computational complexity based on theoretical error estimates for typical classes of solutions and measured computer timing of benchmark problems leads to reliable assesment and conclusions for computational comparisons of the finite element methods.

2- Higher order elements are preferable because they usually need smaller computational effort for the same accuracy of error. Moreover, high degree elements are much more robust (e.g. also with respect to the *locking* problem). The less favourable effectiveness in comparison with lower order elements in extreme situations is significantly outweighed by very good performance in normal situations, especially using a "reasonable" mesh.

3- The concrete conclusions previously listed down in section 4c) are valid for large classes of problems.

4- Our comparisons were based on the performance with respect to the energy norm. We expect the same conclusions for other norms and performance measures although a larger emphasis could be necessary on the used mesh (e.g. dealing with the pollution problem; see *Babuška* and *Oh* [5]).

5- We did not address the question of man power cost. For the comparison of this type in industrial environment we refer to *Barnhart* and *Heisemann* [2] where higher order elements were shown highly preferable.

6- We did not address here the question of the relation of the quality results assessment (see, e.g., *Noor* and *Babuška* [4]) to the degree of the elements. This field is quite new for all methods. Nevertheless hierarchical increase of p leads to a simple and effective error control for all data of engineering interest based on various types of extrapolation (see, e.g., [23,24,26]).

7- We did not address the effectiveness of the finite element treatment of various mathematical formulations of the same problem (for example Kirchhoff and Mindlin-Reissner models for plate etc.). This comparisons will be made in a forthcoming paper.

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Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

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